$$\mathbf{T}=-\begin{Vmatrix}-(\gamma_{,22}+\varphi_{,12}) & (\gamma_{,12}+\varphi_{,22}) & 0\\ (\gamma_{,12}-\varphi_{,11}) & -(\gamma_{,11}+\varphi_{,12}) & 0\\ 0 & 0 & 0\end{Vmatrix},\quad \mathbf{M}=\begin{Vmatrix}0 & 0 & \varphi_{,1}\\ 0 & 0 & \varphi_{,2}\\ 0 & 0 & 0\end{Vmatrix}$$

which agrees completely with the solution given in /3/.

We note that the general solutions of the equations of equilibrium of the theory of asymmetric elasticity obtained here, enable us to solve specific problems.

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A HOLE IN A PLATE, OPTIMAL FOR ITS BIAXIAL EXTENSION - COMPRESSION*

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An outline of a hole of equal strength in an elastic plate loaded at infinity by mutually perpendicular tensile and compressive forces, is obtained. It is shown that under these conditions the hole is bounded by a contour with corners, and its form is found by numerical methods. It turns out that the contour is very close to rectangular, with slightly rounded sides, whose ratio depends on the load.

Let a thin unbounded plate made of a homogeneous, isotropic, linearly elastic material, occupy the region S in the plane of the complex variable $z=x+iy$, and let it be weakened by a hole with an arbitrary, piecewise-smooth boundary Γ enclosing the origin of the Cartesian coordinate system XOY. Specified forces P_x and P_y, $P_y/P_x=\lambda$ act along the axes of this system, and the hole is load-free.

The stress state of the plate is found from the solution of the homogeneous boundary value problem /1/

$$\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=0;\ t\in\Gamma \tag{1}$$

where t is the complex coordinate of any point on the contour, and the Muskhelishvili potentials $\varphi(z)$ and $\psi(z)$ holomorphic in $S+\Gamma$ have the following asymptotic form at infinity:

$$4\varphi(z)=P_x(1+\lambda)z+O(|z|^{-1}) \tag{2}$$
$$2\psi(z)=P_x(\lambda-1)z+O(|z|^{-1})$$

We can also consider problem (1) with the right-hand side $f(t)=-{}^1/_2P_x(1+\lambda)t-{}^1/_2P_x(\lambda-1)\bar{t}$, with respect to potentials decreasing at infinity.

The equal-strength boundary of the hole is determined, as we know /2/, by the condition

$$\sigma_\tau(t)=\text{const},\ t\in\Gamma \tag{3}$$

expressing the constancy of the tangential component of the stress tensor on it (the normal component of this tensor is, according to the boundary condition, equal to zero).

Such a boundary represents the solution of the problem of optimal design of a hole in a plate relative to either of the two optimizing functionals.

A. *The potential energy of the plate deformation functional.* First the integral functional is regularized by subtracting from its density a constant term corresponding to the homogeneous stress field of a solid plate. The functional expresses the weakening effect of a cutout. In /3/ it is shown that when (3) holds, a stationary point (for small variations in the form of Γ) of the functional is reached, provided that the area of the hole is given.

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B. The functional of the maximum of the local Mises criterion of emergence from the elastic state over all points $z \in (S+\Gamma)$

$$F(I_1, I_2) = I_1^2(z) - 3I_2(z) \tag{4}$$

(I_1, I_2 are the first and second invariants of the stress tensor at the point z). It was established in /4, 5/ that (3) represents the necessary condition for a global optimum according to (4).

The problem of the actual determination of the equal strength contour was studied in detail in /2/ under an a priori assumption that the forces specified at infinity are of equal sign. The contour was found to be an ellipse with a ratio of the axes equal to λ. When λ tends to zero of infinity, the ellipse degenerates to a cut parallel to the direction of the load.

The condition of non-negativity of λ is necessary for the existence of an equal strength boundary for an arbitrary number of holes. Indeed, the following relation must hold on such a boundary:

$$x = \beta T(x),\ \beta = \frac{\lambda+1}{\lambda-1};\quad 2x = t + \bar{t},\ t \in \Gamma \tag{5}$$

where $T(\cdot)$ is the integral operator of the double layer potential on Γ. The proof of the identity (5) follows to the latter the proof in /6/, where the case of an axisymmetric system of equal-strength cavities in an elastic space was discussed. From (5) it follows that β is an eigenvalue of the operator T, and the known property of its spectrum implies that /7/, $\beta \geqslant 1$, from which it follows that $\lambda \geqslant 0$. Thus no equal strength contours exist when $\lambda \leqslant 0$.

On the other hand, direct investigation of Problem A leads to a condition of optimality weaker then (3): the only requirement is that the absolute magnitude of $\sigma_\tau(t)$ /3/ should be constant almost everywhere on Γ. This makes it possible to extend, in an interesting manner, the concept of equal strength to the case of $\lambda < 0$ which will be considered below.

Let us call the contour Γ_m on which the following condition almost always holds, the modularly equal-strength (M-equal-strength) condition:

$$|\sigma_\tau(t)| = \text{const},\ t \in \Gamma_m \tag{6}$$

Such a contour can have points at which σ_τ changes the sign discontinuously as a result of a discontinuous change in the tangent unit vector τ, i.e. when the points in question are corner points.

The fundamental local property of the equal strength contour, namely the fact that the Mises yield point is attained at all its points simultaneously for a proportional increase in the load, holds also for the M-equal-strength contour. So far, however, the authors have not yet decided whether the plasticity begins at internal points of the plate, or at the contour. In the case of equal-strength contours the investigation of this fact /4, 5/ is based on the maximum principle, in connected with the function $\operatorname{Re}\varphi'(z)$ harmonics in $(S+\Gamma)$ and converging according to condition (3) to a constant

$$4\operatorname{Re}\varphi'(z) = P_x(1+\lambda);\ z \in S+\Gamma \tag{7}$$

When the contour is of M-equal-strength, condition (7) obviously does not apply.

A numerical procedure for determining the M-equal-strength contour consists, assuming that it exists, of the following. Let the function $\omega_0(\zeta)$ map conformally onto S an auxilliary region D of the complex variable ζ, outside the unit circle γ, with the boundaries and points at infinity matching. With this approach identity (3) simplifies appreciably the problem of determining the equal-strength contour in the regions of arbitrary connectivity: the function $\omega_0(\zeta)$ is a solution (with a prescribed asymptotic form) of the outer dual Dirichlet problem /2/ for the Muskhelishvili potentials

$$\operatorname{Re}[\omega_0(\xi) + \psi_0(\xi)] = 0;\ \psi_0(\zeta) = \psi(\omega_0(\zeta))$$
$$\operatorname{Im}[\omega_0(\xi) - \psi_0(\xi)] = 0;\ \xi \in \gamma$$

which follows from (1) when condition (7) is taken into account. In the case in question, the identity (1) previously differentiated with respect to t, takes a more complex form on γ

$$\xi^2\omega_0'(\xi)\varphi_0'(\xi) - 2\omega_0(\xi)\varphi_0''(\xi) = 2\omega_0'(\xi)\psi_0(\xi) \tag{8}$$

The above condition was used in /8/ to construct the equivalent infinite system of linear relations

$$a_2 - \sum_{k=1}^{\infty} a_k C_{k+1} = P_x(\lambda - 1) \tag{9}$$

$$a_{m+2} = \sum_{k=0}^{m} (m-k+1) C_{m-k+1} a_k + (m+1) \sum_{k=1}^{\infty} C_{m+k+1} \bar{a}_k \tag{10}$$

which was obtained by substituting into (8) the following expansions of the functions $\omega_0(\zeta)$ and $\sigma_\tau(\xi) = \sigma_\tau(\omega_0(\xi))$:

$$\omega_0(\zeta) = C_0\zeta + \sum_{k=1}^{\infty} C_k\zeta^{-k}, \quad \sigma_\tau(\xi) = \sum_{-\infty}^{\infty} a_k\xi^k \tag{11}$$

$$\zeta = Re^{i\theta},\ R \geqslant 1,\ \xi = e^{i\theta},\ \xi \in \gamma,\ a_{-k} = \bar{a}_k$$

and equating to zero the coefficients of positive powers of ξ, the latter expressing the holomorphic character of the right-hand side of (8) in D. The quantity C_0 is a scale multiplier whose value is determined by the area A and does not affect the stress field. By virtue of the symmetry the coefficients C_k with even, and a_k with odd values of k are zero, and the remaining coefficients are real.

The condition of M-equal-strength (6) holds also for γ by virtue of the conformality of the mapping, provided that we regard $\sigma_\tau(\xi)$ as the component of the stress tensor tangent to the circle. Assuming therefore that a_k is known, we obtain from (9), (10), according to condition (6), an infinite system of linear algebraic equations for $\{C_k\}$ determining, in toto, the transformation of γ into the required contour Γ_m. Such an approach was proposed earlier in /5/ for determining the equal-strength boundary under the load $\sigma_\tau(t) \neq \text{const}$.

Let the function $\sigma_\tau(\theta)$, which is even by virtue of the symmetry of the problem, have on its half-period $[0, \pi/2)$ the points $\theta_i\ (i = 1, \ldots, l)$ at which the sign changes. The number and position of these points are defined arbitrarily, and therefore represent independent geometrical parameters of the problem. In the simplest case $l = 1$, so that we have only four such points on Γ_m

$$\sigma_\tau(\theta) = b_0,\ |\theta| < \theta_1,\ \sigma_\tau(\theta) = -b_0,\ \theta_1 < |\theta| < \pi/2 \tag{12}$$

The asymptotic expression (2) also holds in D as $\zeta \to \infty$ /1/. According to the mean value theorem we have, for the harmonic function $\operatorname{Re}\varphi_0'(\zeta)$,

$$a_0 = 4\langle \operatorname{Re}\varphi_0'(\xi)\rangle = b_0\pi^{-1}(4\theta_1 - \pi) = P_x(1 + \lambda) \tag{13}$$

The remaining a_k are found from (12) as Fourier coefficients

$$a_k = 4\pi b_0 k^{-1} \sin 2k\theta_1,\ k = 2, 4, 6, \ldots \tag{14}$$

The quantities $C_k\ (k = 1, 3, 5, \ldots)$ are obtained from the solution of system (10) by virtue of the homogeneity independent of the value of b_0. The latter in turn is determined by substituting $\{C_k\}$ into (9), whose right-hand side has been previously transformed, taking (13) into account, to the form

$$P_x(\lambda - 1) = P_x(\lambda + 1) - 2P_x = 4b_0(4\theta_1 - \pi)/2\pi - 2P_x$$

from which we have

$$b_0 = -\frac{2P_x}{d}; \quad d = \sum_{k=1}^{\infty} a_k C_{k+1} + \frac{4(4\theta_1 - \pi)}{\pi}$$

and finally, from (13) we find $\lambda = a_0/P_x - 1$.

In realizing this scheme of solution system (10) was truncated to $70 \div 90$ terms. The results, some of which are given in Table 1, show that the optimal contour determined by numerical methods differs from a rectangle with a ratio of the sides $\mu = u/v = f_1(\theta_1)$ only in the fact that small segments of the rectilinear boundary adjacent to each corner became rounded, and in the magnitude $\varkappa$ of the angle itself. It is precisely this difference that ensures that σ_τ has no singularities, a fact established in advance by (11–13).

Indeed, when we assume that the singularities of the stress field near the corners have a power asymptotic form, the following relation holds /8/ for its indices $\rho_i\ (i = 1, 2, \ldots, n)$:

$$\sin^2(1 - \rho_i)\,\varkappa = (1 - \rho_i)^2 \sin^2\varkappa \tag{15}$$

The quantity $\varkappa$ is measured by going round the corner in the region S in a clockwise direction.

We have established that when $\pi < \varkappa < 2\pi$, Eq.(15) has simple positive roots less than unity. Thus for a right angle we have $(\varkappa = 3/2\,\pi)\ \rho_1 = 0.4555,\ \rho_2 = 0.0115$.

Further study shows that when $\pi < \varkappa < \varkappa_0$, we have a unique root of the required type. Here $\varkappa_0 \approx 256°$ is the first positive root of the transcendental equation $\varkappa_0 = \operatorname{arctg}\varkappa_0$.

Table 1 shows that $\varkappa < \varkappa_0$ for any λ. The coefficient G_1 of the asymptotic expansion with index ρ_1 is found /8/ from the condition that the load $f(t)$ is orthogonal, and from the homogeneous solution of the biharmonic equation corresponding to ρ_1:

$$2\pi G_1 = \int_{\Gamma_m} f(t)\,\chi_{-1}(t)\,dt$$

In particular, in the case of a square, the function $\chi_1(t)$ does not change its sign after rotation by an angle of $\pi/2$ /9/, while $f(t)$ does change sign; therefore $G_1 = 0$. The same situation obtains for rectangles for any λ.

Using symmetry consideration we can restrict ourselves to the values $\theta_1 \leqslant 45°$, where V is

Table 1

θ_1, deg.	μ	b_0/P_N	$-\lambda$	$\varkappa$, deg.
5	48,6	1,008	0,002	202
15	18,4 (18)	1,09 (1,06)	0,042 (0,046)	216
18	11,33 (11)	1,20 (1,12)	0,115 (0,126)	219
25	5,14 (5)	1,51 (1,47)	0,302 (0,308)	227
30	3,35 (3,26)	1,62 (1,55)	0,311 (0,317)	231
40	1,47	2,57	0,866	244
45	1	2,732 (2,696)	−1	247

Table 2

k	1	2
2	$-0,1667$	$-0,14480$
6	$0,1786\cdot10^{-1}$	$0,1725\cdot10^{-1}$
10	$-0,5682\cdot10^{-2}$	$-0,5740\cdot10^{-2}$
14	$0,2604\cdot10^{-2}$	$0,2699\cdot10^{-2}$
18	$-0,1440\cdot10^{-2}$	$-0,1518\cdot10^{-2}$
22	$0,8917\cdot10^{-3}$	$0,9520\cdot10^{-3}$

the smaller side of the rectangle directed along the Y axis. The brackets contain the values obtained from the solutions due to G.N. Savin /10/ of the problem of uniaxial stretching of a plate with a rectangular hole. The quantity b_0 was determined only for segments of the boundary lying near the middle of the sides, because the stress field near the corners was considerably distorted in /11/ by virtue of retaining only three to four terms in the expansion of $\omega_0(\zeta)$ represented by the Christoffel-Schwarz integral. The same problem for a square hole was solved in /12/ in a different way, namely by solving numerically the integral Sherman equation with preliminary separation of the singularities at the corners. Here good agreement of the results was observed for much large rectilinear segments of the boundary.

Table 2 gives the values of the coefficients of C_k for a square from /11/ (the first column) and results obtained using the method proposed here.

Numerical calculations lead to the assertion that θ_i can change its sign per half-period at most once, otherwise the solution of system (10) will correspond to a non-single-valued mapping, and this is physically meaningless.

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